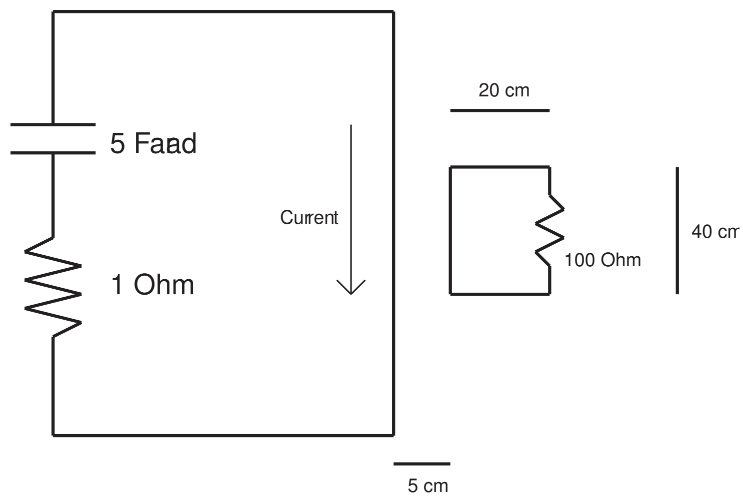
Changing Fields Homework Solutions

1. Consider a very large rectangular circuit with a 5 Farad capacitor charged to 50 Volts and a 1 Ohm resistor. A smaller conducting rectangle is set up 5 cm away from one side of the larger circuit. The smaller rectangle has a length of 40 cm and a width of 20 cm as shown, and has a resistance of 100 Ohms. You may treat the nearby side of the larger circuit as an infinitely long straight wire and ignore all the other wires for the purposes of calculating the magnetic field.



a. State the direction of the magnetic field produced by the current carrying wire through the smaller rectangle.

By the right hand rule, the field will be pointing out of the page.

b. What will happen to the magnetic flux through the smaller rectangle as the capacitor discharges, will it increase or decrease?

As the capacitor discharges, the current will decrease, thus causing the magnitude of the magnetic field around the wire to decrease, as the field strength is given by

$B = \frac{\mu_0 I}{2\pi r}$ where r is the distance from the long straight wire.

Since the field is decreasing everywhere, the flux through the rectangle is decreasing, so less magnetic field lines are pointing out of the page.

c. In what direction will the induced current flow in the small rectangle? Clockwise or counterclockwise?

The induced current be in such a direction as to produce field lines pointing out of the page, in order to oppose the decrease in flux by Lenz's Law. The induced current in the small rectangle must therefore flow counterclockwise.

d. Find the magnetic flux through the small rectangle in terms of the current through the long wire connected to the capacitor. Hint: the field only varies with the distance from the wire, so divide the loop into thin strips of area and integrate to find the total flux.

Let r_1 be the distance from the current carrying wire to the near side of the rectangle and r_2 be the distance to the far side. Let *L* be the length of the rectangle. Divide the rectangle into thin horizontal strips of thickness *dr* and length *L* at each distance *r* from the wire. The total flux through one of the thin strips is then

$$d\phi_m = \mathbf{B}\cdot\mathbf{A} = BLdr\cos\theta = BLdr = \frac{\mu_0 ILdr}{2\pi r}$$

Integrating the flux from r_1 to r_2, the total flux is

$$\phi_m = \int_{r_1}^{r_2} \frac{\mu_0 ILdr}{2\pi r} = \int_{r_1}^{r_2} \frac{\mu_0 ILdr}{2\pi r} = \frac{\mu_0 IL}{2\pi}\int_{r_1}^{r_2}\frac{dr}{r} = \frac{\mu_0 IL}{2\pi}\ln(r)\Big|_{r_1}^{r_2} = \frac{\mu_0 IL}{2\pi}\ln\left(\frac{r_2}{r_1}\right)$$

e. Write the current through the long wire as a function of time, and use this to write the magnetic flux through the small rectangle as a function of time.

The current as a function of time is given by the equation

$I(t) = I_0 e^{-\frac{t}{\tau}} = \frac{V_0}{R}e^{-\frac{t}{RC}}$ where V_0 is the initial potential across the capacitor

Thus $\phi_m(t) = \frac{\mu_0 I(t) L}{2\pi}\ln\left(\frac{r_2}{r_1}\right) = \frac{\mu_0 V_0 L}{2\pi R}\ln\left(\frac{r_2}{r_1}\right)e^{-\frac{t}{RC}}$

f. Write the induced EMF and then current flowing through the small rectangle as a function of time. Graph this function.

By Faraday's law

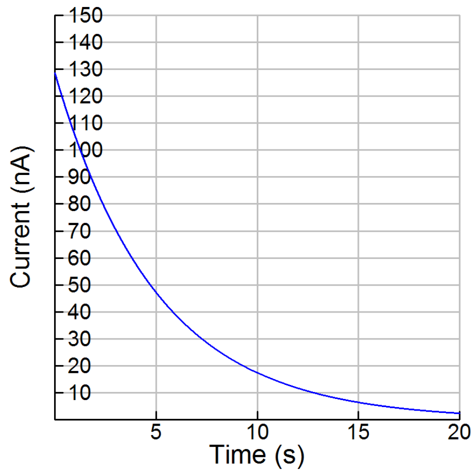
$$V(t) = EMF = -\frac{d\phi_m(t)}{dt} = \frac{d}{dt}\frac{\mu_0 V_0 L}{2\pi R}\ln\left(\frac{r_2}{r_1}\right)e^{-\frac{t}{RC}} = \frac{\mu_0 V_0 L}{2\pi R}\ln\left(\frac{r_2}{r_1}\right)\frac{d}{dt}e^{-\frac{t}{RC}} = -\frac{\mu_0 V_0 L}{2\pi R^2 C}\ln\left(\frac{r_2}{r_1}\right)e^{-\frac{t}{RC}}$$

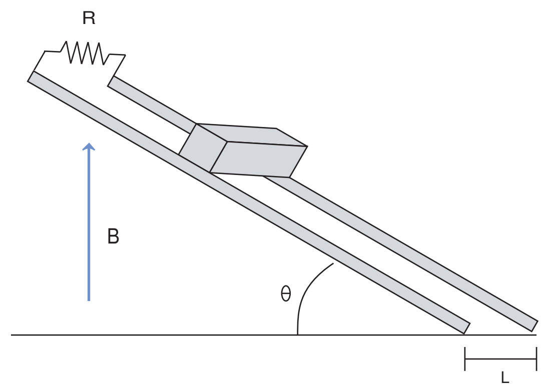
Plugging in constants and known values,

$$V(t) = -\frac{(4\pi\times10^{-7})(50)(0.4)}{2\pi(1)^2(5)}\ln\left(\frac{0.25}{0.05}\right)e^{-\frac{t}{5}} = 1.287\times10^{-6}e^{-\frac{t}{5}}$$

Then by Ohms law $I(t) = \frac{V(t)}{R} = 1.287\times10^{-8}e^{-\frac{t}{5}}\,A = 128.7e^{-\frac{t}{5}}\,nA$

This is an extremely small current measured only in nanoamperes!





2. A rectangular conducting rod of mass m is placed on top of two conducting rails inclined at an angle of θ from the horizontal. A uniform, constant magnetic field B is directed upwards. The rails are connected by a wire with resistance R creating a closed circuit. The force of gravity will cause the rod to slide down the rails with an increasing velocity v.

a. Show that there is a retarding force directed up the incline given by

$$F = \frac{B^2 L^2 v \cos^2 \theta}{R}.$$

If the block has slid down a distance x along the rail, the magnetic flux through the circuit is

$$\phi_m = \mathbf{B} \cdot \mathbf{A} = BA\cos\theta = BLx\cos\theta$$

The change in flux over time is then

$$\frac{d\phi_m}{dt} = \frac{d}{dt} BLx\cos\theta = BL\cos\theta \frac{dx}{dt} = BLv\cos\theta$$

Then the induced EMF must be

$$V = EMF = -\frac{d\phi_m}{dt} = -BLv\cos\theta$$

Where the negative sign is taken to mean that the current opposes the change in flux, and therefore produces a downward pointing magnetic field, so that the current must be flowing out of the page as it goes through the rod. The magnitude of the current is

$$I = \frac{V}{R} = \frac{BLv\cos\theta}{R}$$

This current is perpendicular to the magnetic field, and so produces a magnetic force on the rod directly to the left by the right hand rule, with a magnitude of

$$F_B = \left|I\mathbf{L}\times\mathbf{B}\right| = ILB\sin\left(90°\right) = ILB = \frac{B^2L^2v\cos\theta}{R}$$

The component of this force along the rail is then

$$F_{B_x} = F_B\cos\theta = \frac{B^2L^2v\cos^2\theta}{R}$$ as expected.

b. Show that the terminal speed of the rod is $v = \dfrac{mgR\sin\theta}{B^2L^2\cos^2\theta}$

The force of gravity on the block is $F_G = mg$, and the component of this force directed along the rail is $F_{G_x} = F_G\sin\theta = mg\sin\theta$.

When the component of the magnetic force pulling the rod up the rail is equal to the gravitational force pulling the rod down the rail, the rod will cease to accelerate.