A Tale of Friction

Basic Rollercoaster Physics

Fahrenheit Rollercoaster, Hershey, PA | max height = 121 ft | max speed = 58 mph


PLAY

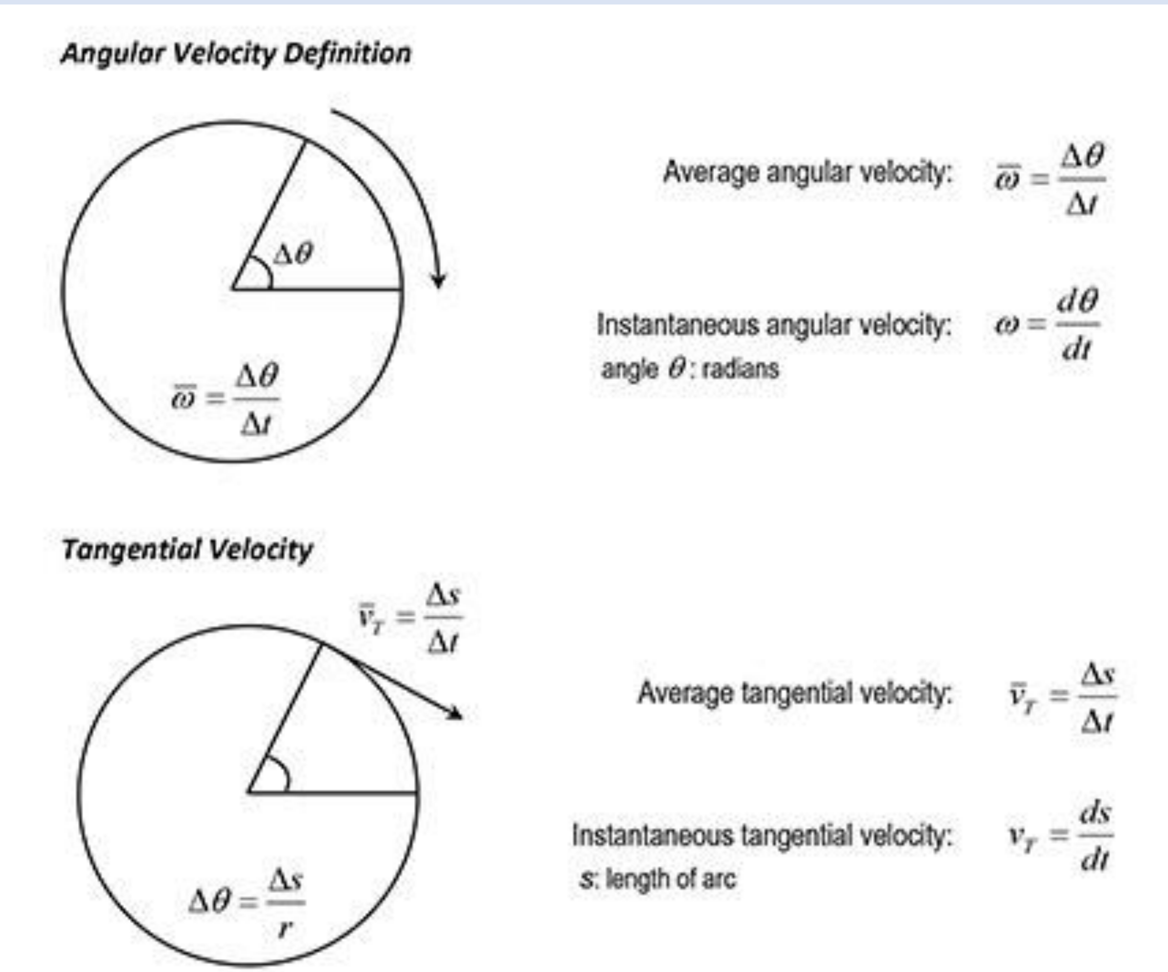

Poltergeist
PLAY


PLAY
RIDE OF STEEL


PLAY

Rotational Movement Kinematics

Similar to how linear velocity is defined, angular velocity is the angle swept by unit of time. Tangential velocity is the equivalent of linear velocity for a particle moving on a circumference.



$$s = \theta \cdot r$$

$$v_T = r\frac{d\theta}{dt}$$ or $$v_T = r \cdot \omega$$

$$\alpha = \frac{d\omega}{dt} \qquad a_T = \frac{dv_T}{dt}$$

$$\alpha = \frac{d^2\theta}{dt^2} \qquad a_T = r\frac{d^2\theta}{dt^2}$$

$$a_{T.} = \alpha \cdot r$$

Rotational Kinetic Energy and Momentum of Inertia of a Rigid Body

- **For a single particle:**

Tangential kinetic energy: $K = \frac{1}{2} m v_T^2$

Rotational kinetic energy: $K = \frac{1}{2} I \omega^2$

Momentum of inertia: $I = m r^2$

- **For a system of particles:**

Momentum of inertia: $I = \sum m r_i^2$

- **For a rigid body:**

Momentum of inertia: $I = \int r^2 dm$

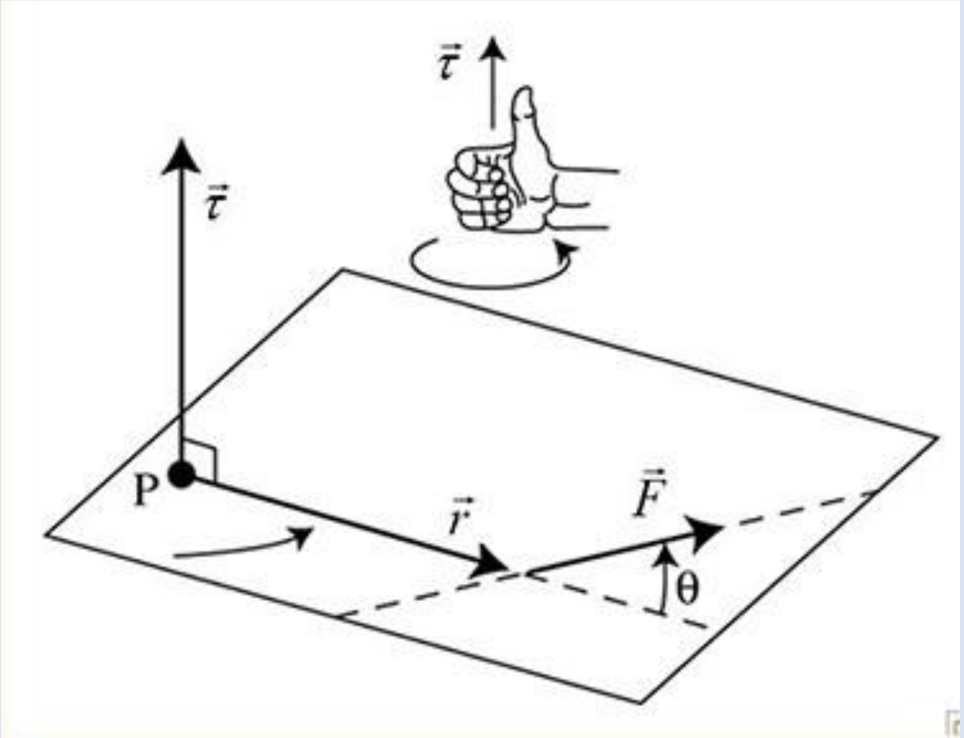
Angular Momentum and Torque of a Rigid Body

- **Law of lever:** $\tau = F \cdot d$
- **Torque:** $\vec{\tau} = \vec{r} \times \vec{F}$

  Magnitude: $\tau = r \cdot F \cdot \sin\theta$

- **Newton's second law:** $\vec{F} = m \cdot \vec{a}$

  $= m \cdot \frac{d\vec{v}}{dt}$



Torque is a measure of how much a force acting on an object causes that object to rotate. It is formally defined as *a vector coming from the special product of the position vector of the point of application of the force, and the force vector*. Its magnitude depends on the angle between position and force vectors. If these vectors are parallel, the torque is zero.

Angular Momentum and Torque of a Rigid Body

- **Linear momentum:** $\vec{P} = m \cdot \vec{v}$
- **Force definition:** $\vec{F} = \frac{d}{dt}(m \cdot \vec{v})$

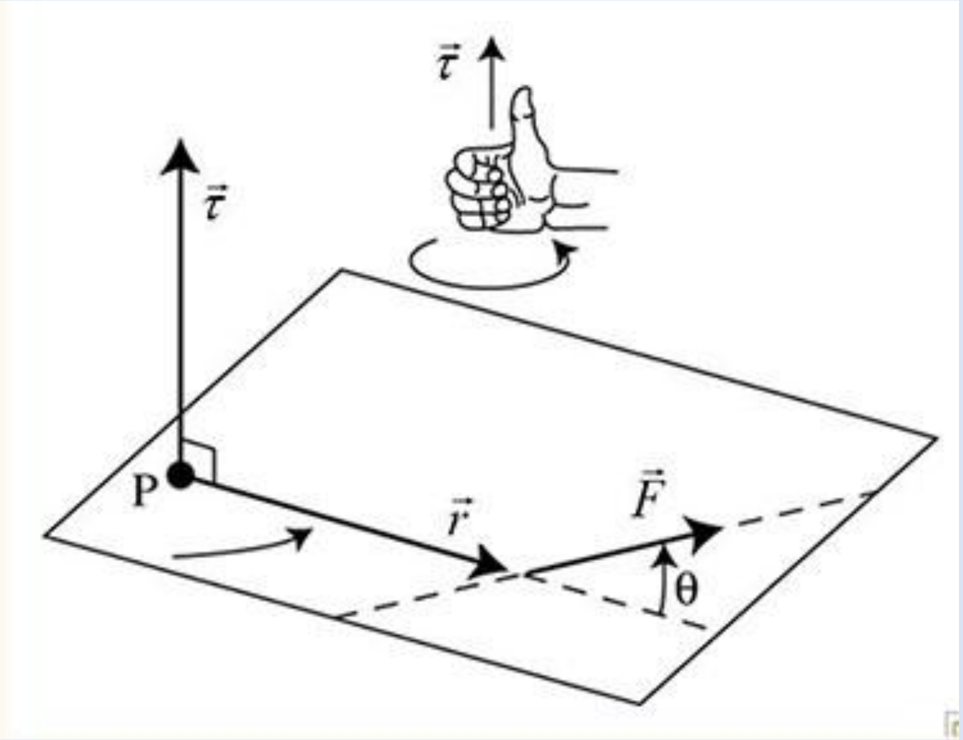
  For m = constant: $= m \cdot \frac{d\vec{v}}{dt}$

  $= m \cdot \vec{a}$

- **Angular momentum:** $\vec{L} = \vec{r} \times \vec{p}$

  $= m \cdot \vec{r} \times \vec{v}$

  $If\ \vec{r} \perp \vec{F}:$ $L = m \cdot r \cdot v_T$



Defining torque (force producing rotation) in a circular movement (*r* constant) as the change in time of the angular moment:

$$\tau = \frac{dL}{dt} = m \cdot r \frac{dv_T}{dt} = m \cdot r \cdot a_T$$

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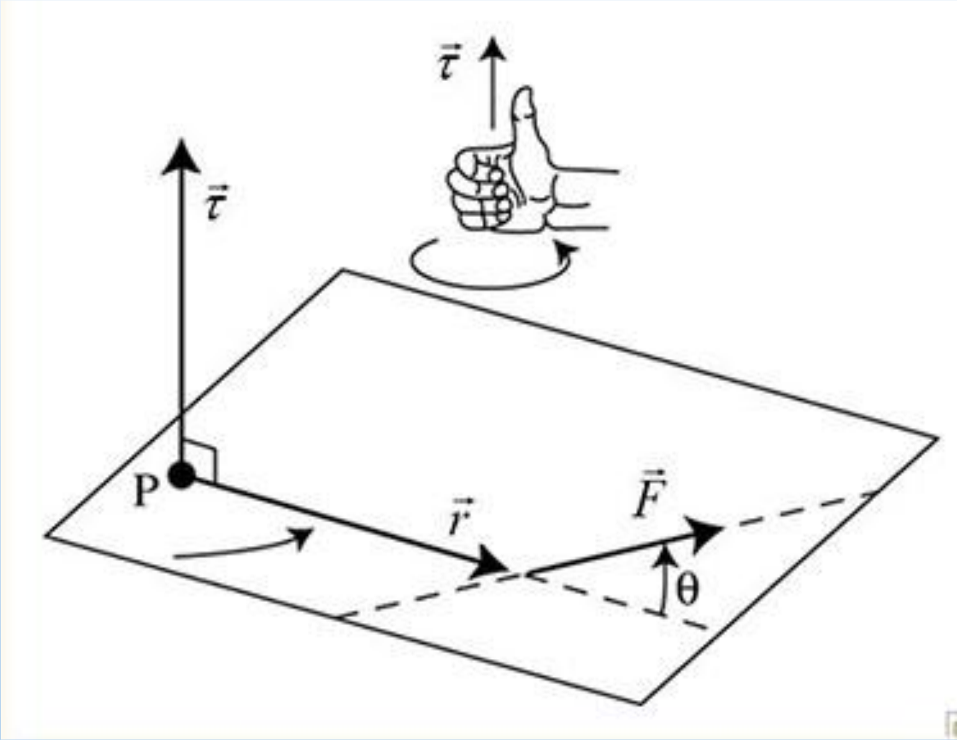


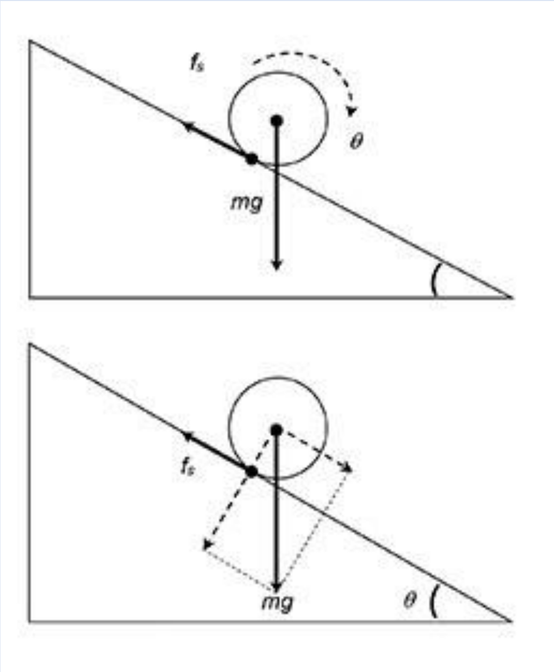
Taking $a_T = \alpha \cdot r$, and making $I = m \cdot r^2$:

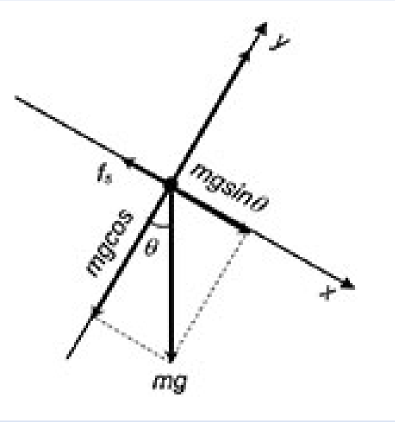
$$\tau = m \cdot r \cdot a_T = m \cdot r^2 \cdot \alpha$$

or

$$\tau = I \cdot \alpha$$

Friction Force for a Rigid Sphere Rolling on an Incline





The sphere rolls because of the torque produced by the friction force f_s and the weight's component parallel to the incline:

$$F = m \cdot a = m \cdot g \cdot \sin\theta - f_s \quad \text{and} \quad \tau = f_s \cdot r = I \cdot \alpha$$

If the sphere's momentum of inertia is $I = 2/5 \cdot m \cdot r^2$ and $\alpha = a/r$:

$$f_s \cdot r = \frac{2}{5} m \cdot r^2 \cdot \frac{a}{r} \quad \text{or} \quad f_s = \frac{2}{5} m \cdot a$$

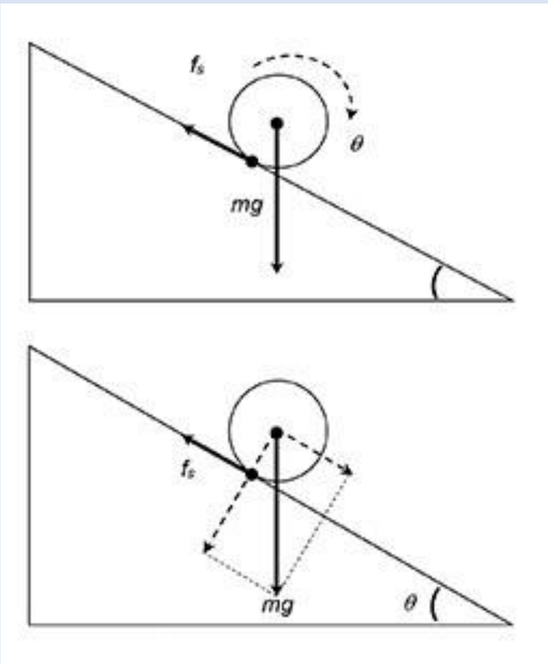
With this value:

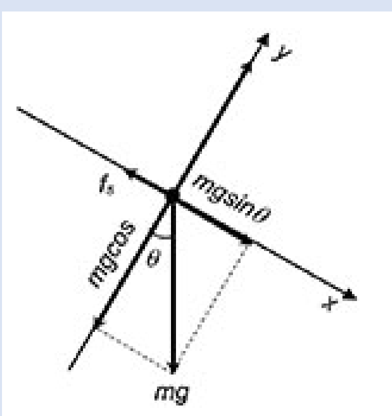
$$m \cdot a = m \cdot g \cdot \sin\theta - \frac{2}{5} m \cdot a$$

Solving for a in the above equation, the acceleration of the sphere rolling on the incline is:

$$a = \frac{5}{7} \cdot g \cdot \sin\theta$$

Friction Force for a Rigid Sphere Rolling on an Incline





Combining: $f_s = \frac{2}{5} m \cdot a$ and $a = \frac{5}{7} \cdot g \cdot \sin\theta$

the static friction force is now:

$$f_s = \frac{2}{7} m \cdot g \cdot \sin\theta$$

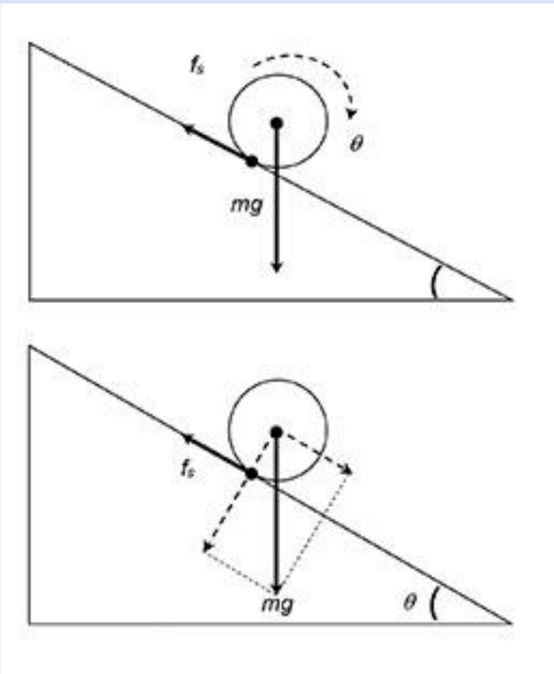
But by definition, the static friction force is proportional to the normal force the body exerts on the surface :

$$f_s = \mu_s \cdot F_n$$

Taking F_n from the free-body diagram:

$$f_s = \mu_s \cdot m \cdot g \cdot \cos\theta$$

Friction Force for a Rigid Sphere Rolling on an Incline

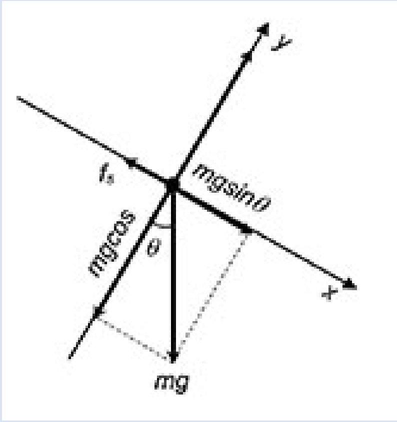


Combining the two expressions for f_s:

$$\mu_s \cdot m \cdot g \cdot \sin\theta = \frac{2}{7} m \cdot g \cdot \sin\theta$$

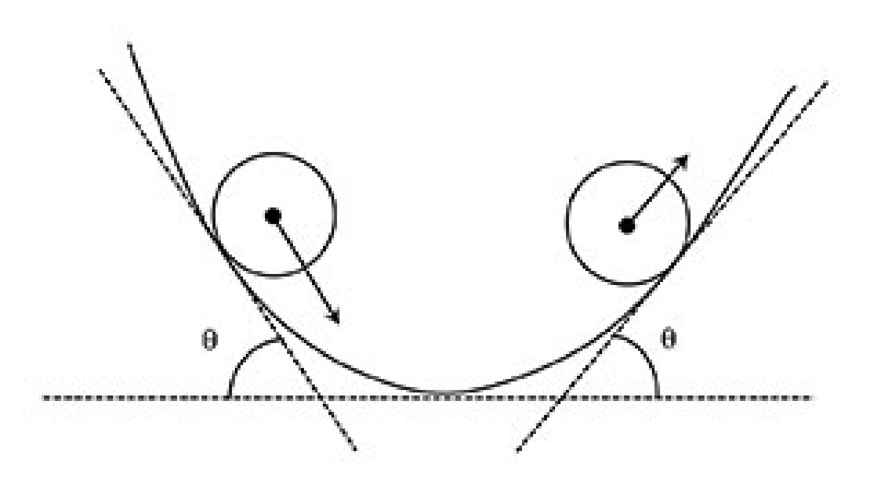
the coefficient of static friction can be expressed as:

$$\mu_s = \frac{2}{7} \tan\theta$$



This expression states that the coefficient of static friction is a function of the incline's angle only, specifically, a function of the slope of this surface.

Friction Force for a Rigid Sphere Rolling on a Variable Slope Path



At any point of a curved path $f(x)$, a tangent line can be visualized as a portion of an incline.

The slope m of this incline is the tangent of the angle between this line and the horizontal, $\tan\theta$.

In calculus, this slope is given by the value of $f'(x)$, the derivative of the function $f(x)$ at that point.

Let $f(x)$ a differentiable function. If:

$$m = \frac{dy}{dx} = f'(x) \quad and \quad m = \tan\theta$$

then: $\tan\theta = f'(x)$

The coefficient of static friction μ_s can be expressed as:

$$\mu_s = \frac{2}{7} f'(x)$$

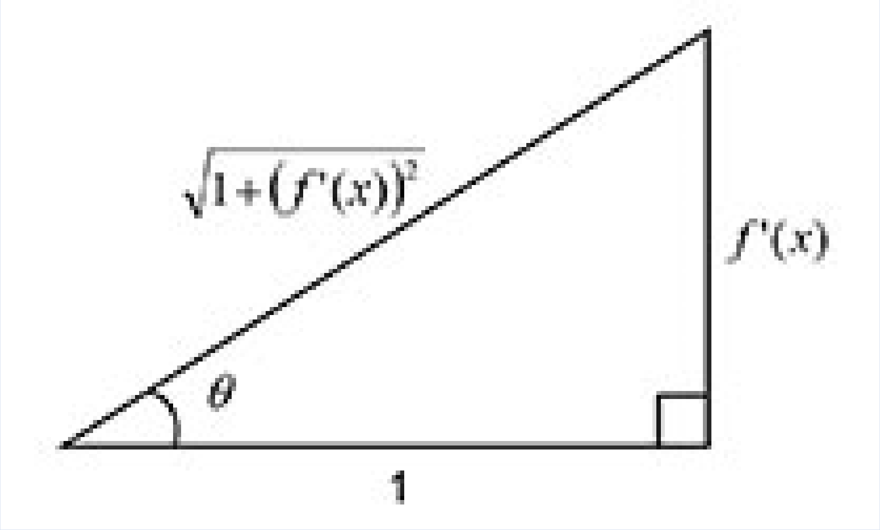
The static friction force f_s is now:

$$f_s = \frac{2}{7} m \cdot g \cdot f'(x) \cdot \cos\theta$$

Friction Force for a Rigid Sphere Rolling on a Variable Slope Path

Because $\tan\,\theta = f\,'(x)$, it is possible to define a right triangle with sides in terms of $f\,'(x)$:

$$\tan\theta = \frac{f'(x)}{1} = \frac{opposite}{adjacent}$$



If: $\theta = \arctan(f'(x))$, then:

$$f_s = \frac{2}{7} m \cdot g \cdot f'(x) \cdot \cos(\arctan(f'(x)))$$

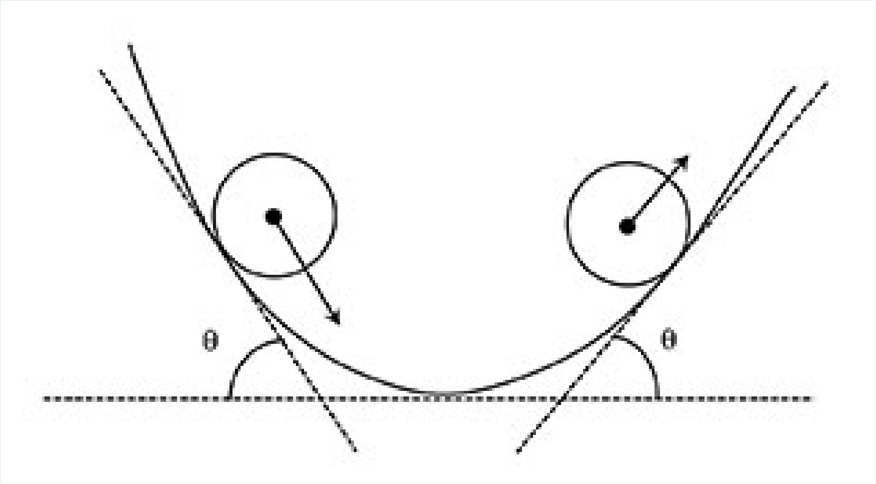
Using basic trigonometry:

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{1}{\sqrt{1+(f'(x))^2}}$$

The static friction force is now: $$f_s = \frac{2}{7} m \cdot g \cdot \frac{f'(x)}{\sqrt{1+(f'(x))^2}}$$

Friction Force for a Rigid Sphere Rolling on a Variable Slope Path

But, something needs to be fixed in this procedure. By definition, the static friction coefficient μ_s must always be positive, while the slope of a path may be positive or negative.



So the required corrections must be:

$$\mu_s = \frac{2}{7}\left| f'(x) \right|$$

$$f_s = \frac{2}{7} m \cdot g \cdot \frac{\left| f'(x) \right|}{\sqrt{1 + (f'(x))^2}}$$

Where: $\left| f'(x) \right|$ denotes the absolute value of the function $f'(x)$

Work-Energy for a Sphere Rolling on a Variable Slope Path with Friction

The **work-energy theorem** states that the mechanical energy (kinetic energy + potential energy) of an isolated system under only conservative forces remains constant:

$$E_f = K_f + U_f = K_i + U_i = E_i$$

$$or$$

$$\Delta E = \Delta K + \Delta U = 0$$

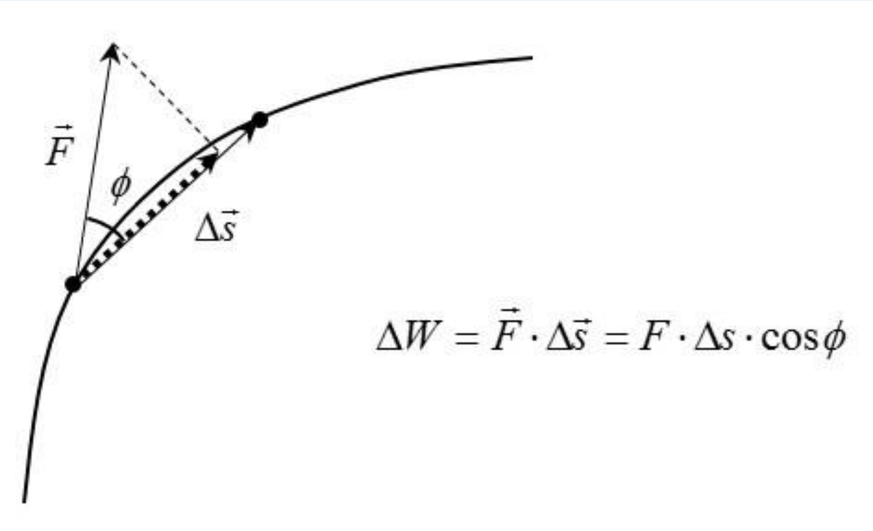
In a system under non-conservative forces, like friction, the work-energy theorem states that work done by these forces is equivalent to the change in the mechanical energy:

$$\Delta W_f = \Delta E = \Delta K + \Delta U$$

Additionally, the work done by non-conservative forces depends on the path or trajectory of the system, or in the time these forces affect the system.

Friction Force for a Rigid Sphere Rolling on a Variable Slope Path

By definition, **mechanical work** is the product of the displacement and the force component along the displacement:



For a variable slope path $y = f(x)$, the work done by the friction f_s over a portion Δs of the path is:

$$\Delta W = f_s \cdot \Delta s$$
$$= \frac{2}{7} m \cdot g \cdot \frac{|f'(x)|}{\sqrt{1+(f'(x))^2}} \cdot \Delta s$$

For a differential portion of the path:

$$dW = \frac{2}{7} m \cdot g \cdot \frac{|f'(x)|}{\sqrt{1+(f'(x))^2}} \cdot ds$$

Friction Force for a Rigid Sphere Rolling on a Variable Slope Path

Expressing ds in terms of the differentials dx and dy, the differential arc can be expressed in terms of the $f'(x)$:

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)(dx)^2} = \sqrt{1 + \left(f'(x)\right)^2} \cdot dx$$

The work along the differential portion of the path can be expressed as:

$$dW = \frac{2}{7} m \cdot g \cdot \frac{\left|f'(x)\right|}{\sqrt{1 + (f'(x))^2}} \cdot ds$$

$$= \frac{2}{7} m \cdot g \cdot \frac{\left|f'(x)\right|}{\sqrt{1 - (f'(x))^2}} \cdot \sqrt{1 - (f'(x))^2} \cdot dx$$

$$dW = \frac{2}{7} m \cdot g \cdot \left|f'(x)\right| \cdot dx$$

Friction Force for a Rigid Sphere Rolling on a Variable Slope Path

Because $dx > 0$, using properties of the absolute value and the definition of differential of a function:

$$dW = \frac{2}{7} m \cdot g \cdot \left| f'(x) \right| \cdot dx$$

$$= \frac{2}{7} m \cdot g \cdot \left| f'(x) \cdot dx \right|$$

$$= \frac{2}{7} m \cdot g \cdot \left| df(x) \right|$$

Friction forces always acts against the movement, so the work done by them must always be negative:

$$dW = -\frac{2}{7} m \cdot g \cdot \left| df(x) \right|$$

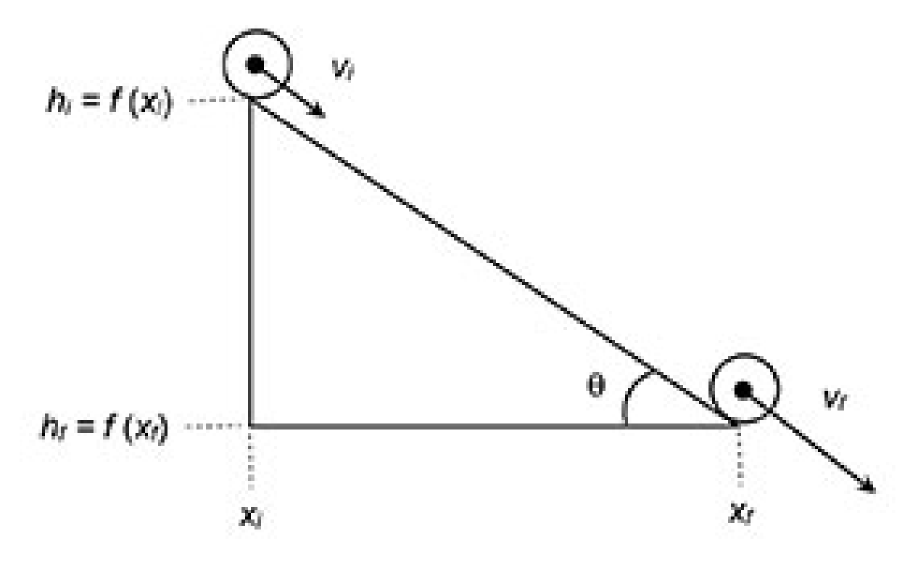
Friction Force for a Rigid Sphere Rolling on a Variable Slope Path

Taking small displacements instead differentials:

$$\Delta W = -\frac{2}{7} m \cdot g \cdot |\Delta f(x)| \qquad \Delta W_f = \Delta K + \Delta U$$

Using this expression in the work-energy theorem:

$$-\frac{2}{7} m \cdot g \cdot |\Delta f(x)| = \tfrac{1}{2} m \cdot v_f^2 - \tfrac{1}{2} m \cdot v_i^2 + m \cdot g \cdot h_f - m \cdot g \cdot h_i$$



This expression relates the work done by friction with the mechanical energy of a sphere rolling **on a little portion of a curved path**.

Visualize this portion as **a little incline**.

Height h is given by the function $f(x)$.

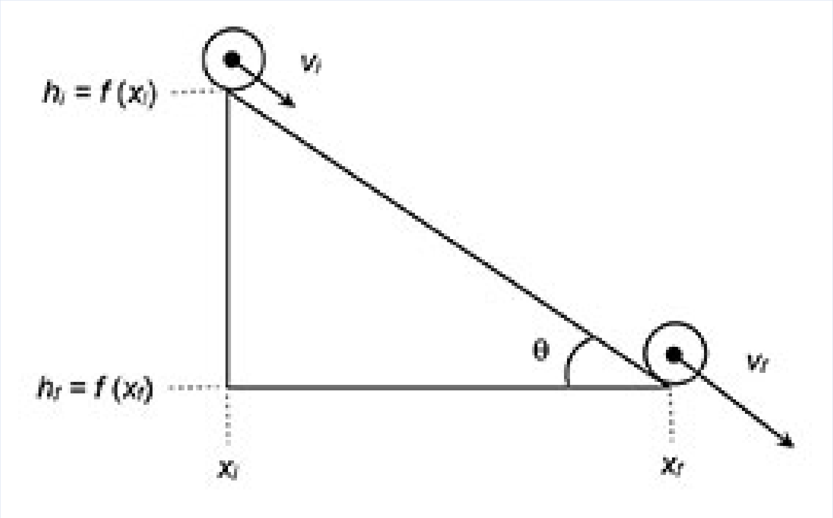
Friction Force for a Rigid Sphere Rolling on a Variable Slope Path

Then, dividing by m:

$$-\frac{2}{7}\cdot g\cdot\left|f(x_f)-f(x_i)\right| = \tfrac{1}{2}\cdot v_f^{\,2} - \tfrac{1}{2}\cdot v_i^{\,2} + g\cdot f(x_f) - g\cdot f(x_i)$$

From this expression, we can determine final velocity at the end of the incline:

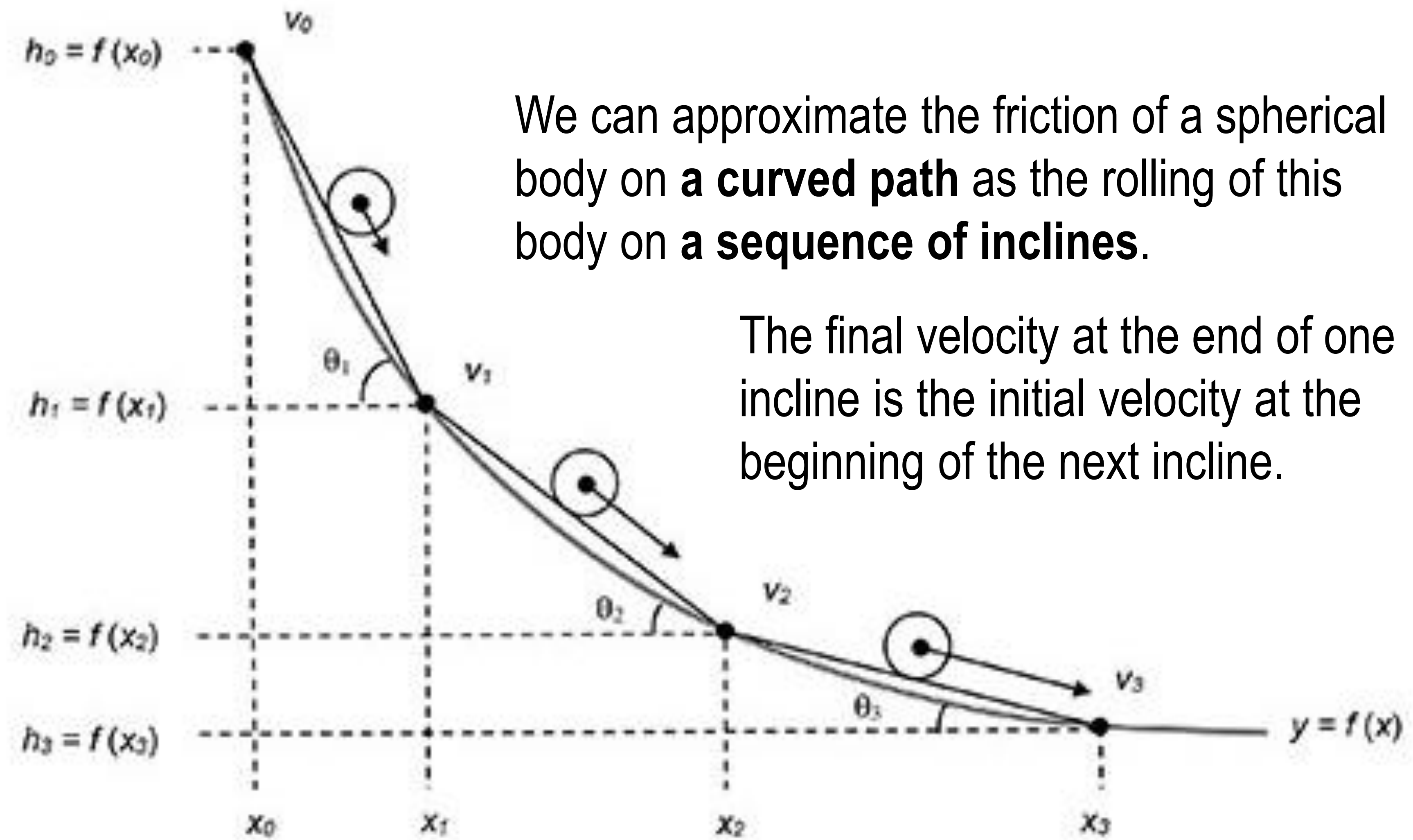
$$v_f = \sqrt{\tfrac{1}{2}\cdot v_i^{\,2} - 2g\cdot\left(f(x_f)-f(x_i)\right) - \frac{4}{7}\cdot g\cdot\left|f(x_f)-f(x_i)\right|}$$



Friction Force for a Rigid Sphere Rolling on a Variable Slope Path

We can approximate the friction of a spherical body on **a curved path** as the rolling of this body on **a sequence of inclines**.

The final velocity at the end of one incline is the initial velocity at the beginning of the next incline.



Are you ready to apply what you have learned?